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ON ELLIPTOGRAPHS, AND ON A MECHANICAL ROCKER FOR DETECTING OSCILLATIONS: BY THOMAS ALEXANDER, M.A.I., Professor of Engineering, Trinity College, Dublin, and A. W. THOMSON, D.Sc., Professor of Engineering, Science College, Poona. To which is added, A MATHEMATICAL INVESTIGATION OF THE FREE PERIOD OF THE ROCKER: BY F. A. TARLETON, LL.D., Professor of Natural Philosophy in the University of Dublin, and Fellow of Trinity College, Dublin.

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XIX.

ON ELLIPTOGRAPHS, AND ON A MECHANICAL ROCKER FOR DETECTING OSCILLATIONS.

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[Read January 26, 1891.]

[COMMUNICATED BY F. A. TARLETON, LL.D.]

The commonest contrivance for mechanically drawing the ellipse is the elliptic trammel which acts on the principle shown on fig. 1. A bar ABC is constrained so that the points A and B, at a fixed distance apart, move in two straight lines at right angles to each other. This is usually accomplished by pivoting the bar at A and B to two blocks sliding in slots OA and OB, crossing each other at right angles at O. As every point in the bar describes an ellipse, the drawing point, usually a lead pencil, may be fixed either at D between A and B, or at C beyond one of them. The ellipses thus described have their centres at O, and their principal axes parallel to the slots. The bar moves precisely as if it were fixed rigidly to the wheel having AB as a diameter, the wheel AB rolling inside another fixed wheel of double its diameter. The radius of the rolling wheel is the algebraic half sum of the semi-diameters of the ellipse described by any point on the bar, while the displacement of the pencil from M, the centre of the rolling wheel, is the algebraic half difference of the semi-diameters of the particular ellipse described by the pencil when in that position.

With the pencil at D, the half sum of the semi-diameters is $OM = AM$, and the half difference is MD; but with the pencil at C, the half difference is OM, and the half sum is MC. It is seldom that the pencil is placed between the sliders A and B; on the diagram the point of the pencil would then be towards you, and the cross require to stand upon legs over the drawing board. In the only trammel that works satisfactorily the cross is small, and sits on the board, and MC is much greater than OM; usually, too, it is a semi-trammel, one arm of the cross being cut short to be out of the way. With a lead pencil it draws nicely semi-ellipses, such as are required for the arches of bridges, and even draws fairly well with the drawing pen when the ellipse differs little from the circle.

One of the pivots at A or B has to be moveable, so as to alter the distance AB, and this introduces a considerable shake, and since the sliding blocks A and B must be of considerable length, so as to pass steadily the crossing at O, the minimum to which AB can be reduced is limited to a considerable size.

In the improved elliptographs which we are about to describe, we use a *fixed* practical and suitable size for the diameter of the *actual* rolling wheel, or distance AB of the sliders apart, and, by the introduction of an additional mechanical contrivance, allow the diameter of the *virtual* rolling wheel to be varied at pleasure, even to the value zero. With such an arrangement it is evident that three studs may be employed, sliding in three symmetrical slots, only one at a time coming to the crossing O.

Besides the imperfections of the trammel already referred to, there is one other rendering it almost practically useless. It cannot draw a clean ink-line with the ordinary drawing pen, which gets broad-side-on at some parts of its path. Syphon pens have been tried, but are too delicate, and can only draw an ink-line of one thickness. Now, it is of great practical importance to be able to draw a line in Indian ink with the ordinary drawing pen, making the line thick or thin at pleasure. To accomplish this, the nibs of the pen must be along the tangent to the ellipse at each point of the path. The chief improvement which we bring forward in this Paper is a *steering gear* for this purpose, in conjunction with the means, already referred to, for altering both parameters of the ellipse through

every variation, without altering the size of the actual rolling circle or span of the actual sliding studs, as the case may be. In short, elliptographs which shall draw ellipses in thick or thin ink-lines, with the ordinary drawing pen, the ellipses ranging from a point to a given size of circle through every variation, including the finite straight line. The apparatus may also be modified to work with a cutting knife or wheel, or with a grinding pad.

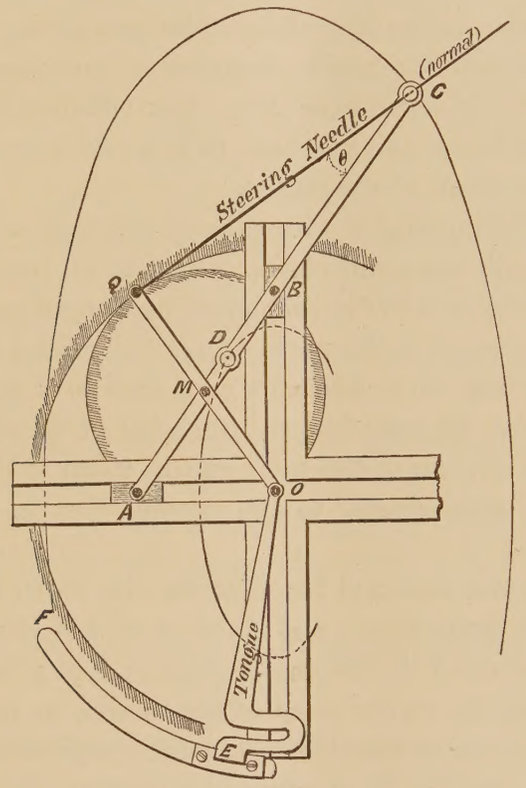

Fig. 1.

OF A STEERING GEAR FOR THE PEN.

The necessity for stearing gear is seen from figure 1. Q, the point of contact of the rolling circle is the instantaneous axis of the motion for the pen C, which is really describing a circle about Q for an instant. Now, if the pen be fixed to the bar at C, with its nibs at right angles to the bar

ABC, as is usually the case, it is clearly moving somewhat side-way-on. If the nibs be nearly closed, the pen will draw a fine line near the extremities of the axes, but only a blurred line at C, and will blot at C if the nibs be well opened. The angle of error is $QCM = \theta$, since the nibs ought to be at right angles to the normal QC for the instant, instead of being fixed at right angles to the bar AC. Since MC and MQ are constant, θ is a maximum when MQC is a right angle, and the sine of the greatest angle of error is $MQ : MC$, or the ratio of the difference to the sum of the diameters. Thus, for an ellipse in which the major diameter is not more than twice the minor diameter, θ could not exceed 20°. The ordinary drawing pen fixed to the bar at D would be quite useless, as it would turn completely broad-side-on in every quadrant of the ellipse.

To adapt our steering gear to the ordinary trammel, a bracket or tongue, EO, is fixed at E with the point of the tongue at O over the centre of the cross. The bar $OMQ = AMB$ is hinged to the tongue at O and to the bar at M, so that Q is always at the point where the rolling circle would be in contact. The pen can turn about its own axis in a socket at C, and a steering needle, with its head hinged to the bar at Q, slips easily through a hole pierced in the stem of the pen, and so steers it like a rudder, with the nibs always at right angles to the needle, that is, always along the tangent to its path.

In the semi-elliptic trammel for drawing the sharp half of the ellipse the tongue may be quite short, and fixed at E, but if the flat half of the ellipse is to be drawn and the tongue be fixed at E, it must then be cranked to admit the neck of the swivel head of the needle, as the needle will pass above the tongue; if the trammel is to draw the complete ellipse, the bracket should be fixed to a stud at E, running in a circular slot, so as to be able to be moved out of the way of the bar when completing the ellipse. The needle, still pivoted at Q, will steer the pen at any point of the bar such as D. By introducing this steering gear, a pen with a dotting wheel, or an engraving or cutting wheel can be used. The application of our steering gear to the common trammel is a little clumsy, but serves our purpose of explaining its action in a convenient way.

OF THE GENERATION OF THE ELLIPSE.

Before leaving figure 1, we may point out that the motion of D is a kind of *planet and anti-satellite* motion; M is like the planet describing a circular orbit round O, while D is the satellite revolving round M, but in an *opposite* direction, and with twice as much angular velocity. The analogue may even be carried further with the pen at D instead of C and steered by a needle pivoted at Q, for then the pen will revolve about its own axis through a right angle for every quadrant which M describes, and in the same direction, but with an angular velocity fluctuating while turning through the right angle. Looking at the motion of D or C as generated in this way, observe that the head of the steering needle has only to be pivoted to a point Q on the radius of M's orbit produced till $OQ = 2OM$.

We have already, then, considered three modes of generating the ellipse, all intimately connected with each other.

1. A point rigidly fixed to a body with two or more studs oscillating in straight slots.
2. A point rigidly fixed to a circular wheel rolling inside another of double its diameter, and—
3. A point describing an anti-satellite motion of double angular velocity round another describing a circular orbit.

OF A KINEMATICAL MODEL TO ILLUSTRATE RANKINE'S METHOD OF THE ELLIPSE OF STRESS.

The manner in which an ellipse is supposed to be generated is often of physical importance. As an example of the anti-satellite method we proceed to give a description of figure 2, which represents the Kinematical Model in the Engineering Laboratory of Trinity College, Dublin, devised by the authors for illustrating "Rankine's Method of the Ellipse of Stress," for uniplanar stress. This model is here described for the first time. It is used for teaching purposes, and consists of a black board (fig. 2) standing upon an easel. On the left of the diagram, ON is a T-square, pivoted to the board at O, and MR a pointer pivoted to the blade of the T-square at M.

At the back of the blade of the T-square, a wheel is fixed to the pivot M, and round this wheel an endless chain is wrapped, the chain also wrapping round a wheel fixed to the board at O, the diameter of the fixed wheel at O being *double* that of the wheel at M. Hence, if the blade of the T-square be caught by the point N, and turned to the right, the pointer MR automatically turns to the left, so that the angle RMN is always 2θ when PON is θ; in other words, the bisector of RMN is always parallel to OP. A thread fastened to the pointer at R, passes through a swivel ring at O, and is kept tight by a plummet. It is easily seen that M moves like a planet round O, while R is its anti-satellite, as described above, and that the head of the steering needle should be pivoted to the blade of the T-square, at a distance from O twice that of M.

CC, the headstock of the T-square, represents the trace on the blackboard of any plane normal to it through O, and the vector of the ellipse, consisting of the segment of the thread RO, represents the stress upon the plane CC in intensity and direction. The position of CC, relative to the plane of greater principal stress is $\theta = PON$, while $\gamma = RON$ is the obliquity of the stress r upon CC. OM and MR are oblique components of r, obtained by supposing the given principal stresses p and q replaced by a pair of equal like principal stresses $\frac{1}{2}(p + q)$, together with a pair of equal unlike principal stresses $\frac{1}{2}(p - q)$.

The model shows clearly the interesting positions of the plane CC; thus, when CC is turned to coincide with AA, the thread RO becomes normal to CC, and of the maximum length p, when CC is turned to coincide with BB, RO is again normal, but of the minimum length q. Again, when CC is so placed that RMO is a right angle, the component of RO, parallel to CC, is a maximum, and, lastly, when CC is turned till MRO is a right angle, then the angle RMO, the obliquity of the stress on CC is a maximum.

The auxiliary diagram, to the right of figure 2, is for solving the inverse, or general problem of uniplanar stress at a point, viz. given the intensity and obliquity of the stress for each of two different positions of the plane CC, to find the intensity and obliquity of the stress on CC when in a given third position. On the auxiliary diagram one T-square fixed to the board represents the T-squares superimposed upon each other for all positions of

CC, and only the pointer *MR* turns. In solving numerical problems upon the stability of earthworks, one linear dimension of the auxiliary diagram represents the known weight of a column of earth, while some other linear dimension represents the required pressure of the earth on a wall, foundation, tunnel roof, or invert. Usually, two angular quantities are known, as $\phi = KON$, the maximum value of γ, being the steepest natural slope

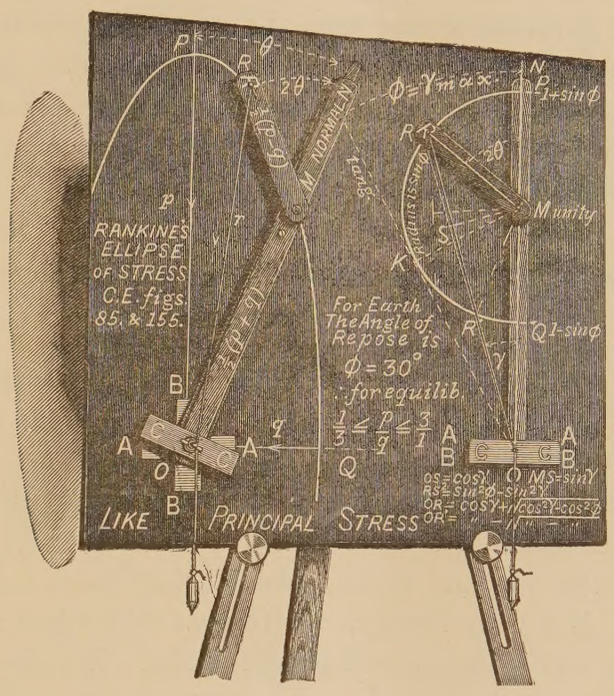

Fig. 2.

of the loose earth, and $\gamma = RON$ the actual slope, perhaps, of the natural free surface. The whole auxiliary diagram is then determinate either graphically or by trigonometry. This auxiliary diagram, in conjunction with trial link polygons, is useful in solving, graphically, problems on the mutual interstability of earth in contact with the roof, invert, base, and side of the thick walls of a tunnel, when the known linear and angular quantities are not given definitely, but merely by pairs of limits between which they must lie.

On the other side of the board a T-square is similarly pivoted to the same pin *O*, and provided with a pointer *MR*, geared in a like way, the difference being that *OM* is shorter than *MR*. It illustrates unlike principal uniplanar stress, and is useful in demonstrating the composition of stresses, such as that of thrust and bending on a pillar, twisting and bending on a crank shaft, the position of the lines of stress in a solid beam, or in the thick hollow cylinder and sphere.

The use of the slots on the easel is to allow the board to be placed so that any required vector of the ellipse may be vertical.

The references on the board are to Rankine's *Civil Engineering*, and numerical examples will be found worked out in detail in an *Elementary Applied Mechanics* by the authors.

OF AN IMPROVED ELLIPTOGRAPH FOR CONTINUOUSLY DESCRIBING THE ELLIPSE OF EVERY ECCENTRICITY.

On figure 3 is shown our improved instrument for continuously describing the ellipse. There is shown a side elevation, a projection of the moveable parts, and a plan of their under side, showing the steering gear. The boss *A* is part of a frame with a heavy foot standing on one side of the drawing board; the frame might conveniently have a number of legs, but they would only serve on the diagram to hide the moving parts. A circular rack, toothed inside, is rigidly fixed to the frame. *BCD* is a cranked strip of metal or *bridge*, having an upper and under deck; it is fixed to the spindle which passes through the boss *A*, and is turned by means of the handle *H*. The bridge may be said to turn about the *main axis* of the instrument. A toothed wheel *M*, having a pitch circle one-half that of the circular rack into which it gears, is pivoted at the point *B* on the upper deck of the bridge. Hence, when the handle is turned, the wheel *M* rolls round inside another of double its diameter, and if the pen were clamped to the wheel *M* directly, at different distances from its centre *B*, a series of concentric coaxial ellipses would be described, all, however, having a common value for the algebriac sum of their semi-axes equal to z, the radius of the wheel *M*. To remedy this, another wheel *N*, exactly like *M*, but not toothed, is

pivoted to a slipper moving in a slot in the lower deck of the bridge *CD*, and which can be clamped by means of the screw *R*, with the axis of the wheel *N* at any distance *x* from the main axis. The under surface of the wheel *M*,

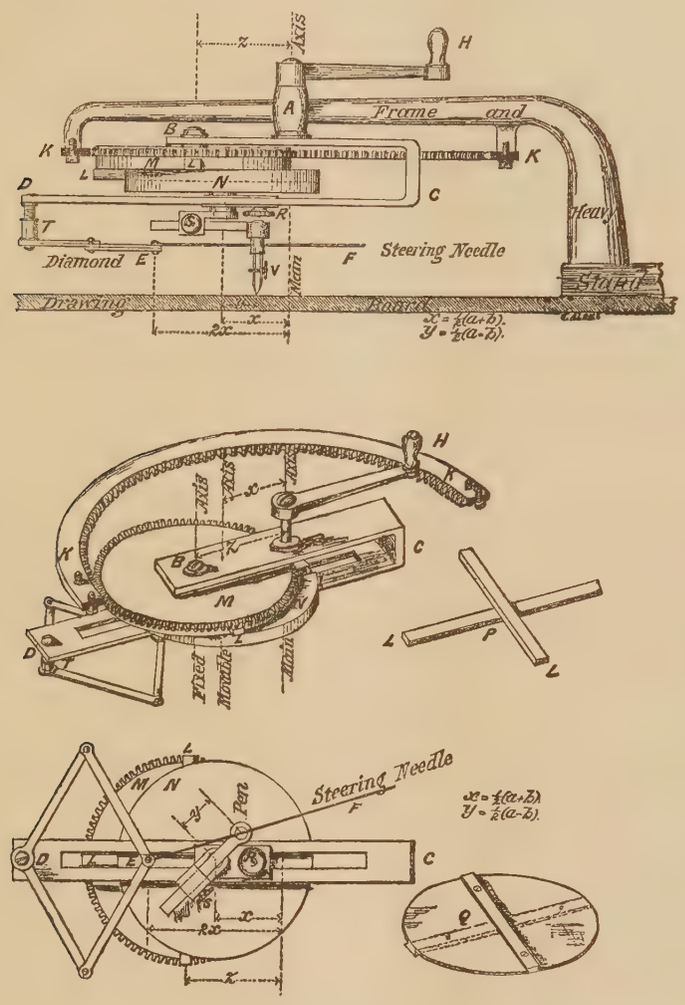

Fig. 3.

and the upper surface of the wheel *N* are just in contact, and they are geared together by what is known as Oldham's coupling, that is, a rigid cross with one arm imbedded in a slot on the under face of *M*, and the other in the upper face of *N*. The cross is shown at *P*, and the ends of its arms

at L. In the model, made by Mr. Yeates, the cross has a diaphragm, as shown at Q, the arms are dovetailed; it is stronger, and works very smoothly. The drawing-pen is shown at V; it is pivoted to a bar that slides in a socket connected to the spindle of the wheel N, and can be clamped by means of the screw S, so that the axis of the pen may be at any displacement y (less than x) from the centre of the wheel N. It will be seen on the diagram that both x and y can be made any value from zero up to a limit fixed by the size of the instrument, so that the pen can describe a complete series of concentric coaxial ellipses with semi-diameters a and b, where

$$x = \tfrac{1}{2}(a + b) \quad \text{and} \quad y = \tfrac{1}{2}(a - b).$$

On a stud at D, the end of the lower deck of the bridge, there is fastened one corner of a stiff articulated diamond frame, and a thimble T can be screwed down on the stud so as to clamp the frame and make it rigid. To the opposite corner E of the frame the head of the steering needle is pivoted, the needle EF passes loosely through a hole pierced in the stem of the pen V. When x and y have been adjusted to any required values by means of graduations engraved on the edge of the slot CD, and on the pen bar passing through the socket S, the thimble T is to be loosened, and the diamond frame distorted till the head of the needle E is set, by eye, at a distance $2x$ from the main axis; the thimble T is then screwed down so as to fix E in that position. The pen will then describe the ellipse continuously with a clear ink line, either thin or thick, or with a dotting or cutting or engraving wheel, no matter how eccentric the ellipse may be. The pen V, to work well, must have a little play up and down in its socket, and advantage can be taken of the stiffness of the diamond frame and steering needle EF to press the pen gently down on the paper. It is also an advantage to have a short tube projecting from the pen V towards F for the needle to slide through, as it holds the pen steady. If it be desired to set the head E of the steering needle more exactly than can be done by eye with the diamond frame, the head of the needle can be pivoted directly to the lower end of the stud DT, the stud being able to slide along DC, and to be clamped at the displacement $2x$ from the main axis by means of a scale engraved on DC, twice as coarse as that by which x,

the displacement of N, is set. DT cannot, however, be a straight vertical stud; as it would come in the way of the pen bar, it ought to be a U-shaped strip standing out horizontally at right angles to DC, so that the pen bar may pass freely inside the U while the U-strip itself passes freely inside the upright part of the stand.

The circular rack may be dispensed with if, instead, a wheel W (fig. 4) be fixed to the boss A, and another wheel, V, of half its diameter, be fixed

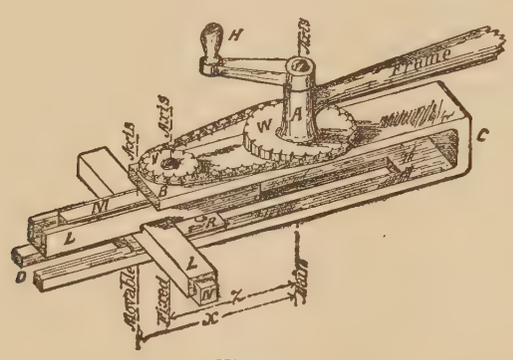

Fig. 4.

to the pivot of the wheel M, a gauged endless chain wrapping round the two wheels. The wheel M need not be toothed; it need not even be a wheel at all, but only a trough to receive the upper arm of the cross.

On figure 4, M and N are shown as rectangular bars, the cross being formed of two troughs, as it can then be made stronger. The wheels W and V, instead of being geared by a gauged chain may themselves be toothed wheels, and geared by a toothed lazy wheel intervening between them, and pivoted to the upper deck BC. It will serve to reverse the motion of V, and double its angular velocity as compared to that of the bridge BCD. Compare figures 4 and 2.*

Another modification is shown in figure 5. A model to this design has been completed by Mr. Yeates—it is the property of the Royal Irish Academy, and lies in the Engineering School of Trinity College, Dublin.

* Mr. Yeates has just succeeded in giving the necessary anti-satellite motion to the moveable axis by having the wheels W and V bevelled and below the deck BC. They are geared by two bevelled wheels on one horizontal shaft between decks and parallel to BC. The shaft can telescope up even till V is below W and Oldham's coupling is dispensed with.

The frame consists of a ring, *RFRF*, standing on four legs, *W*, *W*, *W*, *W*, armed with prickers to pierce the drawing board, so as to hold the frame steady, and at the same time mark the position of the principal axes. A lower ring, *PPPP*, concentric with the upper one runs in four grooved wheels, one on each leg. The bridge *CD*, containing the two wheels *M* and *N*, and the steering gear exactly as in figure 3, is fixed to the lower ring, and turns with it. The wheel *M* is not toothed, and does not gear in a circular rack as in figure 3, nor is it actuated by an endless chain as in figure 4; it is constrained, however, to move exactly as if it were rolling inside a wheel of double its own diameter by being rigidly fixed to the bar *HABK* by means of its pivot *B*; the two points *A* and *K* in the bar being

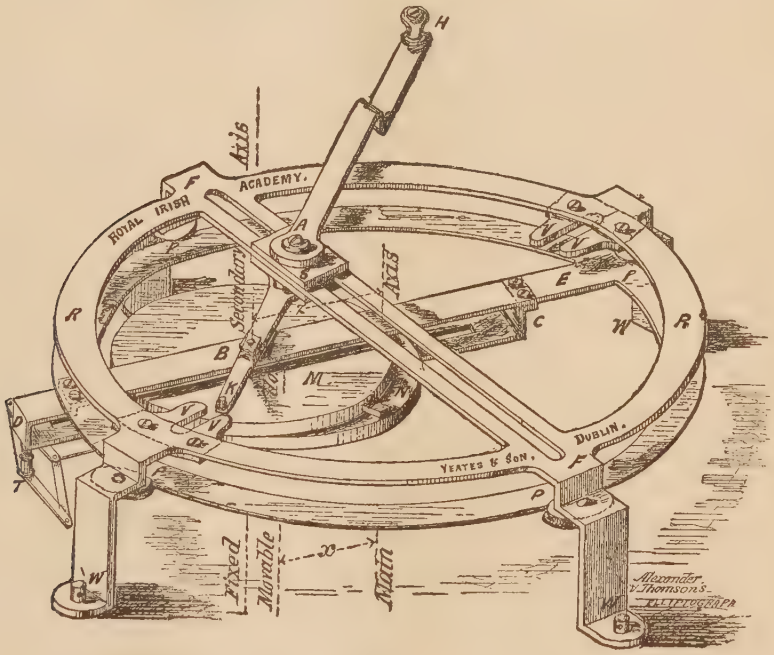

Fig. 5.

constrained to slide in two grooves at right angles to each other, as in figure 1. The pivot *B*, which is midway on the bar between *A* and *K*, is constrained to move in a circle by the lower ring *PP*, and of this additional constraint advantage is taken to allow the pin *K* to disengage from its slot in the middle part of its course, so as to have the other slot *FF*

unbroken. The pin K is low enough to pass under the strips FF. In this way the slipper S has a continuous slot to oscillate in, and the whole mechanism is easily moved by means of the handle H at the end of the bar. The path of H is an ellipse of small eccentricity, which it is just agreeable to the hand to follow. The pin K cannot be altogether dispensed with, for, when the slider S comes to the middle of its stroke, there is a dead point at which the instrument admits of two motions, one elliptic and the other circular. The pin K engages in the small notches, VV, at each end of its stroke, that is, at the middle of the stroke of S.

One difficulty in the construction of the mechanism (fig. 5) is to make the lower ring, which runs in the grooved wheels, to be truly concentric with the cross formed by the slots in the upper ring. If they are not quite concentric, the pin K enters, and leaves the notches VV with a jerk or shock.

OF ANOTHER FORM OF ELLIPTOGRAPH.

An entirely different mechanism for continuously describing the ellipse is shown on figure 6. The same principle is involved. The arms AB and AD turn about a common axis like the hands of a clock, except that they turn in opposite directions, and with equal angular velocity. They are driven by the bevelled wheels and the handle H. E and D are joined by the zig-zag links $EFGD$, and the drawing pen is pivoted at L, the middle point of the bar FG. The pen, consequently, is always midway between E and D, and describes an ellipse for which the longer arm $AB = (a + b)$, and the shorter arm $AD = (a - b)$, the sum and difference of the semi-diameters respectively. If we consider the middle point of the arm AB to be the planet moving round the main axis A, then the pen L is making the anti-satellite motion round the planet with double the angular velocity, as already described. It is evident, too, that E is the proper point to which to pivot the head of the steering needle, being always on the radius of the planet, produced to a distance from the main axis double that of the planet. The steering needle is shown fastened by a little cranked piece to a ring on E; this allows the virtual head of the needle to come close to the pen through a hole in the stem of which it slips freely. If the pen has a little play up and down in its socket, the spring

of the needle will keep it pressed gently down on the paper, and if the needle slide in a tube fixed to the pen, the pen will be kept quite steady. Instead of the bracket frame with its heavy foot, a straddling frame, like that at *S*, may be more convenient, the prickers *p* and *q* serving to hold the frame steady, to mark the major diameter of the ellipse, and to bring its centre over any desired point. With this frame, too, the whole instrument,

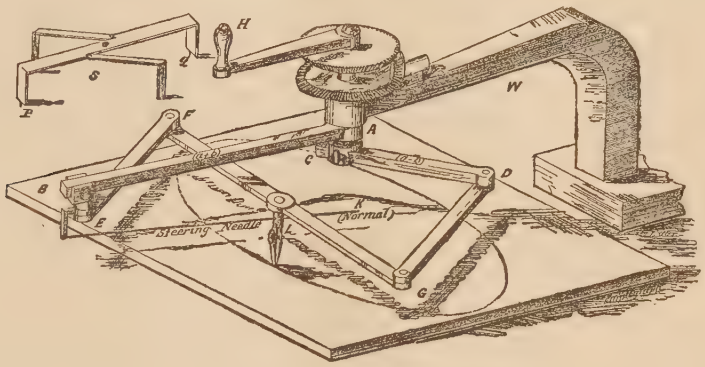

Fig. 6.

including the frame, will close up and pack in a long-shaped portable box. The screws at *C* and *E* enable the lengths set off along the arms to be considerably varied; that of the shorter arm *AD* can be made zero, but that of the longer can only be reduced to a minimum length, sufficient to allow the pen and zig-zag to pass between *EA*. This instrument cannot, therefore, strike ellipses of very great eccentricity, and in that respect is not so general as the forms previously described. This instrument will work with great smoothness, being all *link-work* except the wheels, and it is merely a matter of expense to have the wheels made quite accurately. Instead of bevelled wheels a train of spur wheels may be employed, in which case there may be included change wheels to make the relative counter angular velocities of the two hands other values besides unity, differing slightly from unity, and also a switch can be introduced to reverse the relative motion, that the two hands may turn in one direction. Such an additional train of wheels would be useful for purposes to be explained shortly when we come to describe the more general mechanism of which that shown on figure 6 is only a unique particular example.

OF MECHANISM FOR DESCRIBING THE ELLIPSE WITH TURNING AXES, AND DESCRIBING A CIRCULAR SPIRAL.

A more general mechanism is shown on figure 7, of which the last was the unique particular case. There are two wheels side by side, geared together in any way, either by teeth or a crossed strap, or a train of wheels, so that they may turn with equal angular velocities in *opposite* directions. A and B are pins on sliding blocks clamped in slots cut diametrically on the faces of the two wheels, and are at distances from their centres n and m respectively. $ADCB$ is a lazy tongs with two of its eyes slipped on the pins at A and B, and carrying a pencil at a third eye, such as D or C. The segments into which D divides AB are both to be reckoned positive, while one of the segments, AC, into which C divides AB produced is to be reckoned positive, but the other BC, negative. Instead of the lazy tongs, a pentagraph may be employed, in which case the ratio $p : q$ of the segments to each other can be adjusted to any required value at pleasure. Let the wheels be turned till the slots on their faces are parallel, then both C and D are at the extremity of a principal diameter of an ellipse; C being at the end of a major diameter, and D at that of a minor diameter, with the mechanism arranged, as shown on figure 7, in which m and n, the displacements of A and B, are in opposite directions. It is the more convenient arrangement so as to have the overhang of C as little as possible. By varying m and n, and the ratio $p : q$, the pencil C can be made to describe a complete series of concentric coaxial ellipses, including the straight line circle and point. The common centre is on the line joining the two centres of the wheels, and measured from the point midway between them, the displacement of the centre of the ellipse is engraved on the diagram in terms of m, n, p, and q. Expressions are also engraved on it for the diameters of the ellipse. There is no convenient way of steering the pen.

If the two wheels (fig. 7) be disengaged from each other when the slots are parallel, and if the one wheel be held fixed while the other is turned relative to it through an angle 2θ, and the wheels be then geared to each other again, it is evident that the first wheel must now be turned through

the angle θ to bring the slots parallel again, the other wheel having simultaneously turned back through an angle θ. Hence the effect of turning one wheel through an angle 2θ relative to the other, and reyoking them, is that

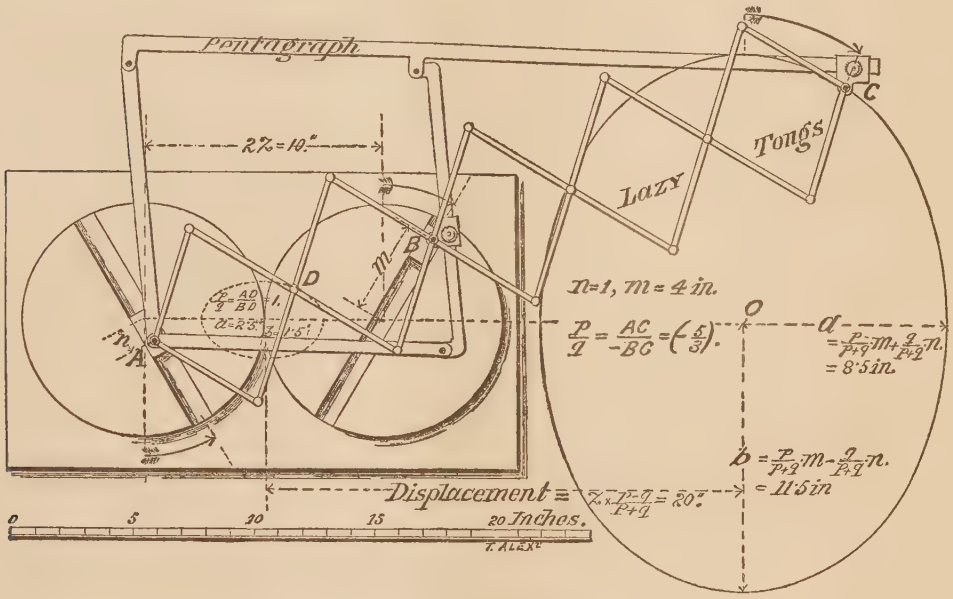

Fig. 7.

the ellipse now described by C has its axis turned through the angle θ with its former direction.

Suppose the two wheels (fig. 7) to be geared together by a train of wheels having a switch by which their motion, with equal velocity, can be made to be in the *same*, or in *contrary* directions at pleasure. Suppose, also, that a *hunting cog-wheel* can be introduced into, and removed again at pleasure from the train, that is, a change wheel which will cause the motion of the two wheels to differ slightly in velocity by a small percentage. Let the wheels be turning in opposite directions with equal velocities, C the while describing the ellipse, let the *hunting cog* be introduced, and the axis of the ellipse which C is describing will now slowly turn round. This motion of C is an imitation of that of the planets. It also resembles the path of the spot of light in Lissajou's figures produced by a double reflection from mirrors on two tuning forks vibrating at right angles to

each other with different amplitudes, the forks being thrown slightly out of tune. This path also is traced by Blackburn's double pendulum.

If the *hunting cog* be removed so as to leave the wheels again revolving contrariwise with equal velocities, and with C describing the ellipse, as in figure 7, and if now the switch be moved so as to make the motion of the wheels *like*, instead of contrariwise, the pencil C will then describe a circle of radius OC, where

$$OC^2 = r^2 = (p^2m^2 + q^2n^2 + 2mnpq \cos \alpha) \div (p^2 + q^2),$$

in which α is the angle between the slots, one on the face of one wheel, and the other on that of the other wheel. These slots are now, of course, always at the same angle to each other. If the *hunting cog* be now introduced, the circle described by C will gradually grow smaller or larger, C, in fact, describing a circular spiral.

Consider figure 7 again, the wheels having equal *unlike* velocities, and C steadily moving in the elliptic path. Now, the switch may be moved so as to make the velocities of the wheels equal and *like* when C is at any point of its path. It follows, then, that the circles which C can describe with this position of the switch may have a radius as great as $b = 11{\cdot}5''$, the major semi-diameter of the ellipse, or as small as $a = 8{\cdot}5''$, its minor semi-diameter, or any mean value depending upon the position of C in its elliptic path at the instant the switch was shifted. Suppose that the switch is shifted while C is at an extremity of the major diameter, the pencil will start off, describing a circle of radius $b = 11{\cdot}5''$, but if the hunting cog be introduced the radius will first of all get smaller and smaller till it is equal to $a = 8{\cdot}5''$, when it will begin to grow larger again, and so on. That is, C is describing an alternately decreasing and increasing circular spiral between the limits 11·5 and 8·5 inches. This, we believe, is an imitation of the path of the moon. Had a originally been zero, that is, had the original path of C been an oscillation in a straight line, then the new path of C would be a complete circular spiral from zero to a maximum, alternately increasing and decreasing, the one continually crossing the other. By only introducing the hunting cog, but not shifting the switch, the straight oscillating path of C would become a looped path continually crossing the centre of, and inscribed in a circle, that is, the straight path would slowly turn round.

APPLICATION TO GRINDING LENSES AND GOVERNING ENGINES.

Suppose a zone to be mapped out on figure 7 by two concentric circles of radii $b = 11{\cdot}5''$ and $a = 8{\cdot}5$. With the hunting cog introduced, and one position of the switch, C will fill up the whole zone with ellipses continually crossing and recrossing, and with the other position of the switch it will fill up the zone with a circular spiral alternately increasing and decreasing, the two spirals crossing and recrossing. When thus furnished with a hunting cog and a switch, we think the apparatus (fig. 7) should prove useful for polishing or grinding zones of lenses and mirrors, or specula. The polishing pad would be fixed to C, and would polish the whole zone with the switch in either position. With the switch kept constantly in the position to cause the pad to fill up the zone with ellipses, the surface would be worked concave, as the ellipses cross oftener, per square area of zone, near the inner edge.

A sensitive mechanical governor is furnished by the mechanism (fig. 7) to regulate machinery where two parts are required to have the same angular velocity, as, for instance, a dynamo and the engine driving it. The two wheels would be driven, one by the dynamo and the other by the engine, with equal velocities when C would describe a vertical straight line or long-shaped ellipse; if one wheel gained upon the other, C would begin to oscillate obliquely, then horizontally, and would either act on the throttle-valve or steam valves of the engine, or throw a resistance into the circuit of the dynamo, and so prevent the racing of the engine, and consequent extinction of all the electric lights, say, when a few happened to be turned out. With the wheels turning in the same direction, the spiral path of C, consequent upon an irregularity of their speeds, could be used to regulate in a like way, and should prove a most delicate mechanical governor for many purposes.

DESCRIPTION OF THE ELLIPSE AND ITS CENTRAL PEDAL BY USING THE CROSSED PARALLELOGRAM.

A favourite way of describing the ellipse is by means of an endless thread passing round the point of a lead pencil and two pegs fixed in the drawing board at the focii of the ellipse, usually two needles. Simple as this arrangement seems, it is rather difficult to neatly draw the ellipse in this way, even with a lead pencil, and still more difficult with the syphon pen. The cord is so ready to slip up on the needles, or off the point of the pencil, neither is the ellipse ever quite the one intended, nor is it placed as it was wished; the cord stretched, or the knot on it slipped, or there was some slight error in the position of the focii. To be quite accurate, too, the pegs and pencil should have the same diameter, so that in any mechanical adaptation of this method, in which the pegs would have a considerable diameter, the series of ellipses it could draw is rather limited, those of small eccentricity being excluded by the pegs coming together, and those of large eccentricity by the pencil being unable to pass between the peg and the band. Other difficulties are a suitable way of altering the length of the band and of keeping it tight, and above all, some manner of steering the pen so as to use ink.

The modification on figure 8 suggests itself, in which two equal separate cords are fastened to the pins A and B, cross each other at E, and have their opposite ends fastened to the extremities of a rod or lath CD, equal in length to the distance between A and B. From symmetry the sum of AE and EB is constant, so that E traces out the ellipse ghk, just as if AEB were one continuous cord, with the advantage, that the cord has not to bend round the pencil point. With one hand the lath BC can be moved, and the cords kept tight, while the other hand is free to hold the pencil against the cords. Further, AD and BC may be rigid bars with advantage; the pencil, too, may be of any diameter if it is held in an external angle at E, like the circle stu. A series of circles like stu, of different diameters, describe a series of concentric coaxial similar ellipses. That is, F, the centre of stu, describes an ellipse of the same eccentricity as ghk, but with

the square of any similar linear dimension greater than that of *stu* in the ratio $Ft + Og : Og$, or the ratio of the sum of the radius of the circle and the minor semi-diameter of the ellipse described by the crossing point of the cords to that semi-diameter. If AD and BC are equal bars instead of cords, half the thickness of the bar is to be included in the radius of the wheel in the above ratio. One focus of the ellipse described by F is at a point on OA produced, and at a distance from O equal to OA, increased in the ratio which is the square root of the above ratio. If a wheel, equal to *stu*, be pivoted at each of these focii, an endless cord will wrap tightly round them, and keep *stu* in its place.

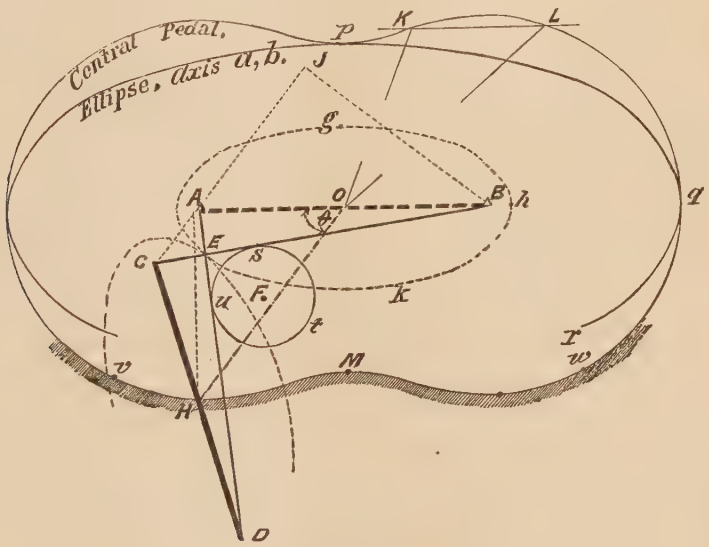

Fig. 8.

It is evident that E is always on an ellipse, of which C and D are the focii, so that H, the middle point of CD, is moving as if it were the centre of an ellipse equal to *ghk*, and rolling upon it. The middle point of OH being on the line bisecting the angle DEB, and OH being at right angles to that bisecting line, it follows that the locus of the middle point of OH is the central pedal of the ellipse *ghk*, and since H is describing the same figure only upon double the scale, then $MHpKLqM$, the locus of H, is the central pedal of the ellipse *pqr*, which is an ellipse double the size of *ghk* and has its major semi-axis $A = BC = AD$, the common length of the cords.

It seems that every point on the rolling ellipse, or rigidly connected to it, is describing some pedal of the ellipse *pqr*. See Greenhill's "Calculus," p. 358. Also when all the pieces *AB*, *CD*, *AD*, and *CB*, are links hinged at *A*, *B*, *C*, and *D*, the link cell is known as a crossed parallelogram.

It is very easy to construct the central pedal, using a little slip of cardboard *CD* with a hole for the point of the pencil at *H*; *AD* and *BC* being two threads tied to pins at *A* and *B* and knotted to holes in the card at *C* and *D*. If the lengths of the threads $AD = BC$ be four-thirds of that of the card $CD = AB$, then the pedal will resemble that on figure 8. It will be fiddle-shaped, the portion *vHMw* being like a switchback railway in which *v*, *M*, and *w* are on one level.

We have dwelt at some length upon the arrangement in figure 8, having found, by experiment in the first instance, a curious case of stable and unstable equilibrium depending upon the proportion of the length of the bar *CD* and that of the two crossed cords supporting it, which we think is new or at least not very well known.

OF AN INTERESTING EXAMPLE OF UNSTABLE EQUILIBRIUM.

If a uniform heavy bar *CD*, figure 8, be hung up by two equal crossed-cords *AD* and *BC* to two pins *A* and *B* fixed horizontally in a smooth wall at a distance apart equal to the length of the bar, then the bar will hang horizontally in stable equilibrium only if the cords be longer than the diagonal of the square described upon the bar. If the cords be shorter than that diagonal, the equilibrium is unstable when the bar is horizontal, and there are two positions of stable equilibrium when the bar has swung to one or other side so as to bring the crossed cords at right angles to each other, as shown by figure 8 where *BED* is a right angle. The experiment can be readily made with the lath of a window-blind and two threads hung to two ordinary pins pricked into the wall paper. The lath should project an inch or two beyond *C* and *D* where the threads are tied to it to prevent the lath swinging out from the wall and the threads uncrossing, or one of the threads should be double with the other passing between the pair. With

long threads the lath will be found to swing about the horizontal position like a pendulum; with threads much shorter than the diagonal of the square on $AB = CD$, the lath will refuse to remain horizontal, but will swing about the position shown in figure 8 where the threads are at right angles. If the length of the threads be only a little less than that diagonal the lath will swing *past* the horizontal two or three times alternately into the position CD in figure 8, and a similar position to the right.

We give a simple geometrical proof. Join C to A and produce it, and drop BJ perpendicular upon it. Put the length of the cords $AD = BC = a$, and of the rod $CD = AB = ea = \sqrt{a^2 - b^2}$. Then a and b are the diameters of the ellipse ghk or semi-diameters of the ellipse pqr, and e is their common eccentricity. Put θ equal to the angle AOH, and observing that CA is parallel to OH, we have—

$$BJ = BA \sin BAJ = ea \sin \theta, \text{ and } CJ^2 = BC^2 - BJ^2, \text{ or } CJ = a\sqrt{1 - e^2 \sin^2\theta}.$$

And if y be the distance of H from AB, then $y = OH \sin \theta$. But $OH = CJ$, each being an average of the sum of CA and DB. Hence

$$y = a \sin \theta \sqrt{1 - e^2 \sin^2 \theta} = 1/ea \,.\, BJ \,.\, CJ.$$

That is, the depth of H the centre of gravity of the rod below the horizontal AB is proportional to the product $BJ \,.\, CJ$. Since J is a right angle, that product is greatest when $BJ = CJ$, that is when $JCB = 45°$, when by symmetry CAD is also 45°, so that AEC is a right angle. Now, if AgB is greater than a right angle, E must be a point on the ellipse to one or other side of gOM before AEB can be a right angle, in which case H, the centre of gravity of the rod is as low as possible, the rod sloping, and the cords at right angles. But the condition that AgB is obtuse, together with the facts that Ag is half the length of a cord and AB the length of the rod, gives an equivalent condition that the length of the cord is less than that of the diagonal of the square described upon the rod.

In this, case, then, H, the centre of gravity of the rod, is describing the switch back curve shown shaded on figure 8. If, instead of being cords, AD and BC are equal uniform heavy bars, the common centre of gravity of the

three bars AD, BC, and CD is always on OH at a distance from O, which is a constant fraction of OH; hence, the common centre of gravity of these three bars is describing the same switchback path as H only in miniature, and the position of equilibrium of the three bars is the same, namely, with the crossed bars at right angles.

It follows from this mechanical problem, as a corollary, that the central pedal of an ellipse is of two characters, fiddle-shaped like that on figure 8 or concave everywhere inside, depending upon whether the pair of lines joining an extremity of the minor axis to the focii make an obtuse or acute angle.

The same criterion will determine whether the equilibrium is stable or unstable in the case of a pair of equal elliptic cylinders, or of a pair of equal spheroids, one of the spheroids rigidly fixed with its axis of symmetry vertical, and the ohter placed in a similar position upon the top of it.

An interesting property of the fiddle-shaped central elliptic pedal, immediately following from the manner in which we generated it, is that if a straight line be drawn parallel to its major axis cutting the pedal in four points, the sum of the squares of the four vectors from the centre to those points is constant and equal to twice the square upon the semi-major diameter. Or if K and L are two of the four points lying on one side, then

$$OK^2 + OL^2 = 4Oh^2 = a^2 = Oq^2 = AD^2.$$

In Thomson and Tait's "Natural Philosophy," second edition, vol. i., p. 399, there are given four figures, two of which show a horizontal bar supported by two crossed fibres and loaded in different ways with a gyroscope. In these the fibres are *long*, and it is correctly assumed that the bar will hang horizontally. Some new interesting cases could now be added to those, namely, the bar supported with *short* crossed fibres hanging in the stable and unstable positions we have already described, and loaded with the gyroscope in different positions; and the rocker about to be described could also be similarly loaded; and furnish instructive mechanical experiments.

OF A MECHANICAL ROCKER.

In the mechanism figure 8, let CD be a uniform *heavy* bar, and the cords AD, BC be replaced by rigid bars of inappreciable weight compared to CD, and let the bars be freely jointed at their ends, and then suppose the whole system turned upside down. The path of the centre of gravity of the system—that is the path of H, the centre of the heavy rod—will be the same switchback curve $vHMw$, only turned upside down. For the motion of the system in the plane, M will be the lowest point and H one of the two highest points of the path of the centre of gravity of CD. Hence the equilibrium will be stable when the heavy bar CD is horizontal, and unstable when the crossed bars AD and BC are at right angles. The same remarks apply exactly if the crossed rods AD and BC are themselves uniformly heavy, as the common centre of gravity of the system is describing a path exactly similar to that of H, but on a smaller scale. The system of bars is unstable for motion out of the plane, but can be rendered stable by having hinged joints at A, B, C, and D, or by having all the bars double.

This last arrangement is shown on fig. 9, which constitutes a mechanical rocker or oscillator. In the model made for the Royal Irish Academy by Mr. Yeates, DC is a uniform brass platform, and the legs are double steel bars pivoted to the platform at D and C, and to a wooden stand at A and B. The model may be described as being like an ordinary camp stool without a pin at the crossing of the legs, and bound together by a brass strap at the top equal in length to the stride of the feet, which are also tied together. The height of the stool is a little less than the breadth, which is equivalent to the rule already given. The feet, instead of being bound together, may sit in little pits as at α and β. The model is stable, with the platform horizontal, the stand being also horizontal. It would constitute a sensitive level; the spirit level would be attached to the platform and a slight displacement from the level of the stand cause the platform to make a large displacement. If, with the finger you depress one side of the platform, say D, it will press up against the

finger so long as DEC is obtuse; and if you release it, it will recover its horizontal position, pass it on the other side, and continue to rock about the horizontal position, in which it will finally settle at rest. If the side of the platform be depressed till the legs are at right angles, and then a little further still, the whole will topple over sideways, and lie flat on the stand; or if the platform projects at the end, as at D, it will topple ever till the end of the platform touches the stand. By having the joints a little stiffened, and using both hands, the rocker with care may be balanced in the position of unstable equilibrium with the legs at right angles. A sufficient blow or shock given to the stand will destroy the equilibrium, when the platform will either fall completely over or fall back horizontal, in which position it will settle after rocking a while. The particular model, of which figure 8 is a representative, beats seconds almost exactly.

It is most remarkable that a weight placed on the middle of the platform increases the stability, while a cylinder placed resting on the crossed legs decreases it, since a small displacement of the platform from the horizontal *raises* the weight at its centre up the fiddle-shaped pedal, while it depresses the cylinder at E to a lower point of the elliptic path of E. This is an exception to the general rule of structures that the lower a load is placed upon it the greater the increase of stability.

APPLICATION OF THE ROCKER TO DETECTING THE PERIOD OF EARTHQUAKES AND OTHER SHOCKS.

One use of the pattern shown in the centre of figure 9, where the platform is rigid and heavy, and the legs slender and supple, is to act as a mechanical starter for earthquake registering machines, such as those used by Professors Milne and Gray for automatically registering the direction of earthquake shocks, and the magnitude of the different components, and which are established throughout Japan and other countries subject to gentle earthquakes.

The starter is set in the position of unstable equilibrium with the legs at right angles; there may be a cross bar on the leg to prevent it falling

over, so that when disturbed it must fall back to the horizontal. If the earthquake shock come in the direction AB, it will destroy the equilibrium, and the instruments are started by electrical contact being made. If the shock come at right angles to AB, the supple legs eat up the vibrations and the equilibrium is not destroyed. Two starters will stand at right angles, and so sort out the shocks according to direction. Experiments with the model can be made by making the joints a little stiff, and placing it lengthwise on a table in the position of unstable equilibrium; when jolting the table in one direction starts it off, but jolting it in the other fails to start it—only making it rattle.

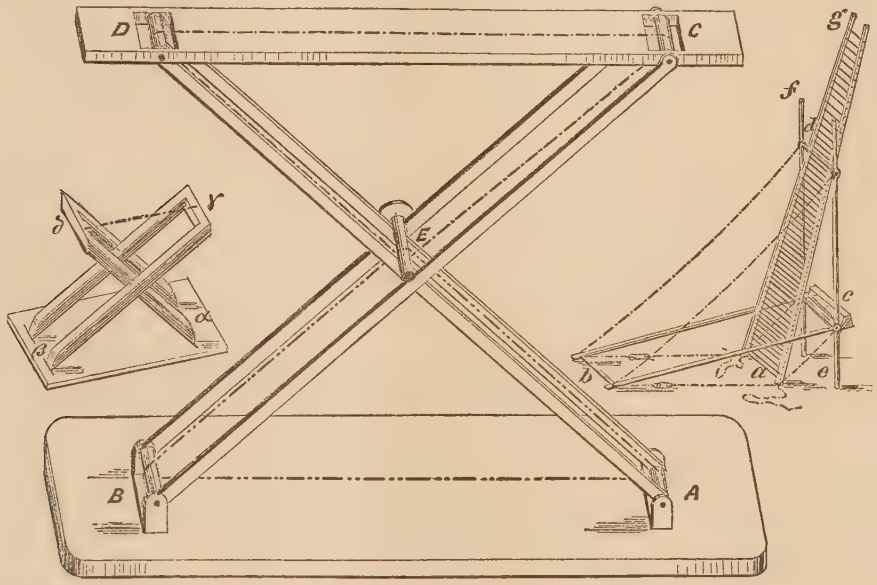

Fig. 9.

The pattern shown at α, β, γ, δ, fig. 9, is for detecting the natural period of an oscillation such as that of an earthquake shock. In this pattern the legs are rigid horseshoe-bars, so as not to eat up vibrations but be sensitive to shock. They have knife-edge feet, which sit in notches on the stand, and are bound together at the top by a slight horizontal wire. The horseshoe-bars can be made cheaply in numbers all alike, and by placing the feet in notches closer or further apart, and by binding them at

the top by wires of different lengths, they can be made to have different free periods of oscillation. A metal stand accommodates a number of these rockers, all tuned to beat different rates when rocking, and lying in different directions in azimuth.

The plate itself is fixed to a heavy masonry pillar sunk in the earth, such as are commonly used for earthquake machines. When a shock comes, all the rockers will rock; but those which have their free periods in unison, or in certain proportions to those of the earthquake shock, will rock with an *increasing amplitude* till they topple over altogether, while the others will settle back in the horizontal position. In this way a permanent record is to be obtained for sifting out the period of the earthquake shock or of its components. Sets of pendulums, used for this purpose, signally failed, as they could not be got to register the result, either as to time or direction, although they gave some information if actually watched during the shock. The rocker can also be used as an escapement for clocks, or any purpose for which a pendulum is used, and should prove more compact and satisfactory, chiefly in that it does not require a point of suspension, but can stand on a plane.

APPLICATION OF THE ROCKER TO SCAFFOLDING, AND AS A FIRE ESCAPE.

The remarkable fact that a horizontal platform, borne by crossed legs, is in stable equilibrium, provided the height does not exceed the width, should prove useful in the construction and erection of light scaffolding for many purposes, to which we can only allude in this Paper.

Thus DC, figure 9, may be a light platform projecting past D and C a certain length at each side, where shear legs DA and CB are jointed to it. The whole will fold flat together, in which way it can be brought to the site readily in a cart or barrow, where it is to be laid down flat, and the feet of the shears pulled together by means of a block and tackle, the platform meanwhile rising up horizontally, and being quite safe, although it rock a little. The rocking can be checked by guy ropes hanging from each end, and fastened to weights on the ground, or one projecting end of the platform may be held on the ground, when the other end will rise up

first. Two men could easily erect such scaffolding for the purpose of pointing or painting. One of the shear legs may conveniently be a ladder, and a workman, or load of tiles, &c., can be got upon the platform by simply pulling one projecting end of the platform to the ground by means of a guy rope, placing the man or load upon it, and hoisting them up by pulling the guy rope attached to the other end. The energy expended in doing this is simply that required to lift the load, because the common centre of gravity of the platform and shear legs is *sensibly moving in a horizontal straight line,* that is in the switchback path *vHMw*, figure 8. By fixing the ends of the guy ropes, the platform can be made secure and rigid in many positions, and by slacking out the feet it can be lowered so as to reach a large part of a wall without shifting. In a like way shears could be used for erecting lofty telegraph posts, or railway signal posts, the post acting itself the part of the platform.

Suppose that the central part of figure 9 is a platform such as we have been describing, then instead of guy ropes the following more permanent arrangement could be used for moving and manipulating it. A cylinder is placed resting on the legs where they cross, and furnished with discs at its ends to prevent lateral motion. Similar cylinders are placed at *A*, *B*, *D*, and *C*, and an endless chain or cord *CED*, shown by a dot and dash line, is wrapped round *D*, *C*, and *E*. It is to wrap many times round *E* and *C*, or be gauged in a way to prevent slipping on these cylinders. Another endless chain is wrapped round *A*, *B*, and *E*, and secured to the cylinders *A* and *E*; it is shown by the double dot and dash line. From the remarks on the mathematics of figure 8, the cylinder *E* should, like the circle *stu*, strictly speaking be in the external angle of the legs, but by suitably warping the two endless chains in figure 9, they will both remain tight for all the motion required to bring either projecting end of the platform to the ground. Suppose now that the two chains, figure 9, are wrapped round the cylinder *E*, then suppose that the cylinders *A* and *C* are each provided with a winch which can be clamped or locked. Then if both the winches *A* and *C* are locked the structure is rigid, for in order to move the cylinder, *E* would have to turn in both directions at once. Now by unlocking the winch at *A* and turning it, the end *C* of the platform can be

brought down. Hence a workman may hoist himself and a load thus: he will unlock the winch at *A*, and by turning it bring the projecting end near *C* to the ground. He will then lock the winch *A*, place his load on the projecting end at *C*, go himself upon the platform, unlock the winch at *C*, and by winding it, raise himself and load to any height he may wish; and that without the expenditure of much energy, as the centre of gravity of the apparatus is moving sensibly in a horizontal straight line.

Another application of the principle of the rocker is to that of a fire-escape as shown on the right hand of figure 9. One of the crossed legs is a ladder *ad*, which is produced on to *g*, and tapers so as to have its centre of gravity as low as possible, and the other crossed leg is loaded at *c* to bring its centre of gravity into a corresponding position. The whole can be brought folded up flat and laid flat on the ground projecting at right angles to the building if there is plenty of room, at an angle if the width of the street is not enough, and nearly parallel to it if there is only a narrow lane. Then *a* and *b* are to be pulled together by a block and tackle, and if the end *a* of the ladder be held on the ground, the weight of one man being sufficient, then the top *g* of the ladder will rear up till the stilts *fe* are vertical and their ends *e* come to the ground, when the ladder will be at a suitable gentle slope, and can be used simply as a ladder if there be no hurry nor danger in passing up and down from flames issuing from lower window. The last is a real danger, for only the other day firemen in London were badly scorched by flames from a lower window. If there be a rung or two at intervals on *dc* then a short board may be shot out horizontally to any window lower than *d*, the board being secured to the rung on *dc* and a rung of the ladder. Observe that the ladder does not require anything for its top *g* to rest against, but can be made quite secure by tightening the guys *db* and *ca* and tying their ends at *b* and *a* to cleets. Another way of using this fire-escape, so as to give greater expedition and avoid the danger of smoke and flames on ascending and descending the ladder, is as follows. Instead of tying the guys *db* and *ca* tight, a crew of firemen will stand at *b* having hold of the ends of both of these guys and keeping them tight. They are themselves well back from *e*, the base of the wall of the burning house. By pulling on the guy *db* and paying out the

guy *ac* they can fetch the top of the ladder *g* close to the ground behind *b*, when the stilts *ef* will have turned over till they are again vertical, but with the end *f* on the ground. The part *df* of the stilts will be provided with hand-holds by which the active crew of firemen can climb up on to the ladder at *d* and thence to *g*, while the land crew at *b*, by pulling on the guy *ca* and slacking the other, will rapidly hoist them up into the position *g* shown on figure 9. Any people rescued by the active crew, especially if they be weak or faint, will be made fast to the top of the ladder *g* by lashings there for the purpose; and on a signal, the land crew at *b*, by pulling one guy and slacking the other, will bring them rapidly close to the ground behind *b*, instantly away from the smoke and heat, where they will untie them and hoist up the ladder again. In lowering and hoisting the ladder, the only work done is that of lifting and lowering the load of people on the end *g* of the ladder, since the common centre of gravity of the machine itself is moving in a horizontal line, sensibly straight, namely a gentle switchback path like that of *vMw*, figure 8.

The stilts *cb* and *fe* would be made of hollow steel tubing, and the guys of wire rope, or chain, so as not to burn.

An arrangement, too, like the cylinder *E* with the endless band *ABE* running round *E*, and two similar cylinders at *A* and *B* could be adopted instead of the guys. Instead of *ABE* a pair of endless steel chains would be employed, the ladder passing between them, and the chains going round the projecting ends of the cylinder *E*, to which they would be secured. The cylinder *B* would be provided with a winch for actuating the whole, and would have a locking pawl.

OF A LINK CELL FORMED BY A COMBINATION OF TWO CROSSED PARALLELOGRAMS.

Two crossed parallelograms yoked together by a common side are shown on figure 10. For symmetry both pairs of crossed bars are equal, but variety can be introduced by having one pair of a different length from the other. The sides are necessarily all equal. The combined cell consists of three equal short links, *AB*, *CD*, and *EF*, and of four equal longer links,

AD, BC, AE, BF, all freely jointed together at the ends so as to allow free motion in the plane of the cell. G, H, and K are the centres of the short links, and M, N, of two of the longer links. First, suppose the cell to be hinged to the plane at K and that G is constrained to move in a straight slot KG, and H in a straight slot KH. Then, for any position of G and H, as we already proved while describing figure 8, we have figure 10.

$$KG^2 = a^2(1 - e^2 \sin^2\theta) \text{ and } KH^2 = a^2(1 - e^2 \sin^2\phi),$$

where a is the common length of the four long links, and ea that of the three short links. Now if the slots KG and KH are at right angles, $\theta = BKG$ and $\phi = BKH$ are complements, when we have $GH^2 = KG^2 + KH^2 = a^2(2-e^2)$ a constant quantity for every pair of simultaneous positions of G and H, each in its own slot, and AB turning about the pivoted point K, it follows that if we pivot G and H to the plane at the fixed distance $a\sqrt{2-e^2}$ apart, releasing K, then K will be constrained to move in the arc ab of the semicircle described on GH, since GKH is always a right angle.

A model can easily be made with strips of cardboard and rings or eylets. The numbers 1, 2, 3 at B show the order in which the links are piled on each other, 1 being the nearest to the observer. The rectangular border round figure 10, represents the edge of a piece of stiff cardboard forming the frame of the model, to which two tongues of cardboard are rigidly fixed each by a pair of eylets tightly closed. G and H are hinged to the ends of the tongues by eylets gently closed, as are also the eylets A, B, C, D, E, F, forming the joints. A good proportion is to make the four long strips each four inches, and the three short ones three inches each, then GH will be $4\sqrt{(2-9/16)} = 4{\cdot}8$ inches nearly. If CD and EF be placed in one straight line, G and H will be at the proper distance apart, their positions can be marked through on the tongues and the eylets closed. Small holes should be pierced at the centre of each bar, M, N, K, &c., for the point of the pencil or pricker to pass through. The locus of the central points are all circular as long as the pairs of long bars are crossed, G and H are points, the extreme case of a circle; K's locus is ab, a finite part of the semicircle GKH, and M, N, and M', N' all move on the same

circle about O, the diameter being a, the length of the longer bar. A neat brass model was made by Mr. Yeates, and is the property of the Royal Irish Academy.

The large central diagram, figure 10, is only slightly disturbed from a symmetrical position where CD and EF are in one horizontal straight line, and AB vertical. While in this position, the cell is readily moved through a short range on each side by moving the point A back and forth. If A be moved to the right till K comes to b the end of its path, the cell will then be in the position shown at Z. It is at a *dead point*, and admits of two distinct

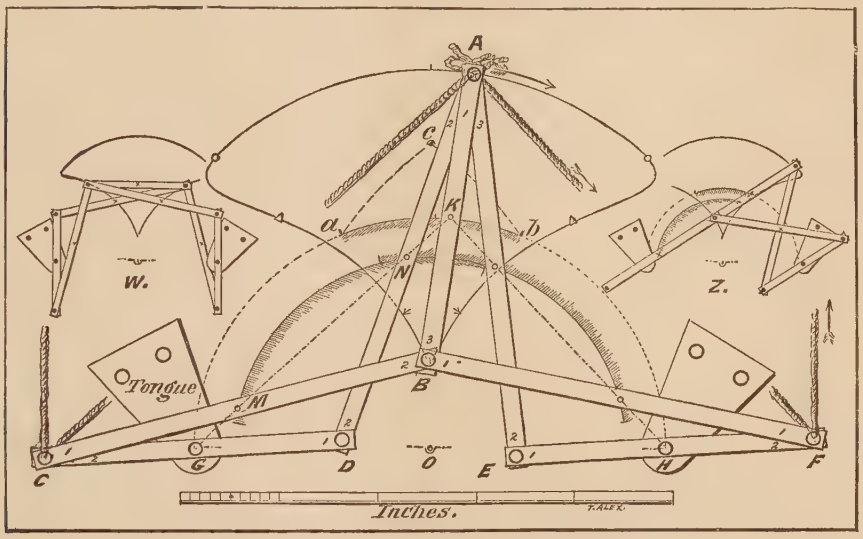

Fig. 10.

motions. To continue its legitimate motion, with both parallelograms still crossed, CD and EF are to be simultaneously turned through a small angle in the *same* direction as they were turning before reaching the dead point. After passing the dead point Z, the motion may be continued by driving one point A, and the cell will come into the new symmetrical position W. In this position K has again come to the middle of its path ba on the back journey, and one of the long bars has just begun to slip under the tongue at H. By still continuing the motion of the cell in the same direction, and passing another dead point, the cell comes back into

the first symmetrical position as on the large central diagram, only with AB turned upside down; past this it cannot move in that direction. From the original position the cell can turn in the opposite direction till AB is again upside down. With this legitimate motion, the cell can turn through one complete cycle during which K rocks once back and forth on ab; M rocks once back and forth on the shaded arc of the circle shown at Z, and A and B each go completely round the pear-shaped locus, a great portion of the crown of which is sensibly a circle about B, and a great portion of one check sensibly a circle about G. The polar equation to the pear-shaped locus, with O as pole, and putting r for the vector and θ for the angle it makes with the vertical, is

$$\cos\theta = \frac{3a^2 - e^2a^2 - 2r^2}{2\sqrt{a^2 - r^2}\sqrt{2a^2 - e^2a^2}}$$

$$= \frac{39 - 2r^2}{2\sqrt{23}\sqrt{16 - r^2}},$$

when $a = 4$ and $ea = 3$.

During the legitimate motion of the cell, EF and CD are always parallel; and in each of the symmetrical positions (central diagram and W) AB is at right angles to them.

When the cell is at the dead point, as at Z, we may, instead of turning EF and CD in the same direction to continue the legitimate motion, get past the dead point by turning them in *opposite* directions when the left parallelogram will open out and be no longer crossed. The locus of K will now be the circular arc bc about H as a centre. Its locus is the symmetrical arc ac if the left parallelogram be crossed and the right one open. When both parallelograms are open, K remains stationary at c. For every possible motion of the cell, K is always on the triangle abc whose sides are circular arcs.

PRACTICAL APPLICATION OF THE DOUBLE-CROSSED CELL.

The middle points of one pair of long-crossed links are M and N (fig. 10); let M' and N' be the middle points of the other pair. Then, if A be moved at a uniform rate to the right, M and N part from each other at a rate which is decreasing while M' and N' approach each other at an accelerating rate. If there be a pair of radius bars from M and N to O, serving to open and shut one cock, while another pair from M' and N' serve simultaneously to shut and open another cock; then, by moving A at a uniform rate, the one cock will shut at a decreasing rate, while the other opens at an increasing rate. Such an arrangement might be useful for regulating the flow of steam.

Suppose a cord to be fastened to the ring A (fig. 10), one end of the cord being reeved through the ring at F (or through a pully block attached to the ring), and taken back in the direction of the arrow, the other end of the cord in a similar way being reeved through the ring at C, and taken back. Then, by alternately pulling one cord, and slackening the other after the manner of tiller ropes, the link-cell can be turned a short way back and forth on each side of the central symmetrical position. The distance on each side must fall short of the dead position of Z. In this way two rudders whose stems are at G and H, can be simultaneously turned with the rudders always parallel by pulling on the tiller ropes. There is, too, a considerable gain of power in pulling on one of the tiller ropes, for when the ring at F has moved any distance in the direction of the arrow, a point on the rope at the arrow has moved three or four times as far.

If both tiller ropes be pulled at once, but that at F pulled harder than the other, and if the pulling be kept up till the cell comes into the dead position, as at diagram Z (fig. 10), then, up to that point, both CD and EF have been turning contrary-clock-wise. If the pulling be still further kept up, EF will go on turning contrary-clock-wise, but the left parallelogram will open out, and CD will suddenly turn clock-wise. Hence, by continuously pulling on the tiller rope at F, there being a counter-strain on the other tiller rope, CD will first turn one way, and then suddenly turn

the other. Such an action might be useful as a starting trigger; thus, a line attached to C would be pulled in one direction, and then suddenly be released.

The model of the cell (fig. 10) can be pivoted at G and H to the drawing board by means of two drawing pins with G and H at different distances apart, when the limited motion of the cell can be studied. Points other than G and H can be pivoted, and afford by inspection interesting results.

ON A PROPOSED MECHANISM FOR CONTINUOUSLY DESCRIBING BICIRCULAR QUARTICS.

One part of the mechanism is Peaucellier's cell, and consists of four equal links, hinged to form a rhombus; two equal longer links are hinged, one to each of two opposite corners of the rhombus, and have their other extremities hinged together. This last hinge is provided with a pricker to stick into the drawing board at a point to be called the *pole*. The two corners of the rhombus which are always in one straight line with the pole are provided with pencil points, and the two lateral corners of the rhombus may conveniently be provided with castors to run on the drawing board. If the cell be moved about in any way, the two pencils describe a pair of what are called *inverse curves*. The tops of the pivots of the two lateral joints have slots, like the nick on the head of a screw nail, only deeper; and a straight lath is placed horizontally sitting with its lower edge in those two slots which are able to freely slide along the lath. If the lath be constrained to move so as always to be tangential to an ellipse, the *inverse curves* described by the pair of pencil points will now be a certain class of Bicircular Quartics. The lath should rest its weight on three castors, one at one end, and two on a cross bar at the other end.

The second part of the mechanism to constrain the lath will be like the elliptograph shown on figure 6. It has a frame with a heavy foot standing to one side of the drawing board. Two horizontal arms hang over the board, and turn about a common centre like the hands of a clock, only in opposite directions, with equal angular velocities. Instead of the zig-zag link, $EFGD$ (fig. 6), a rhombus formed of four equal links will have its two opposite

corners pivoted, one to an extremity of each hand, so that that diagonal of the rhombus will, like the steering needle (fig. 6), be always normal to an ellipse, and consequently the other diagonal of the rhombus always tangential to the ellipse, since the intersection of those diagonals is describing the ellipse, just as the pencil is doing in figure 6. The lower ends of the pivots of those two corners of the rhombus are to have slots which will grasp the upper edge of the lath, being free to slide along it. This will produce the proper constraint on the lath, and the two pencil points of the Peaucellier's cell will now describe Bicircular Quartics. Any one of the four slots, two of which grasp the upper and two the under edge of the lath, may be pivoted to the lath to make the whole combination steadier.

If the pole be chosen inside the ellipse, the apparatus will go completely round each pencil, drawing a closed curve. If the pole be on the periphery of the ellipse, the lath can swing round, being a tangent to the ellipse at every point in succession of a large arc of the ellipse, at the ends of each of which the two pencils come together, and between them describe a closed curve, including a horse-shoe-shaped area. It is all the real part of the curve, for if the pencils could cross they would go back, only having exchanged paths. When the pole is outside the ellipse, the instrument can be set in two distinct positions. In one of them the pole and the centre of the ellipse are always on one side of the lath; in the other they are always on opposite sides of the lath. In each of those two positions the instrument can start with the two pencils together, and make an excursion in which they separate and again come together. By placing the instrument in the two positions in succession all the real parts of the curve are drawn.

MATHEMATICAL INVESTIGATION OF THE FREE PERIOD OF THE MECHANICAL ROCKER.

By F. A. TARLETON, LL.D., Professor of Natural Philosophy in the University of Dublin.

Two equal bars, AB and CD, are attached to two other equal bars, AD and BC, by smooth hinges, the bars AD and BC being the diagonals of the quadrilateral of which AB and CD are opposite sides. If the bar AB be fixed horizontally, let it be required to find the position or positions of equilibrium of the system, and to determine in each case whether the equilibrium be stable.

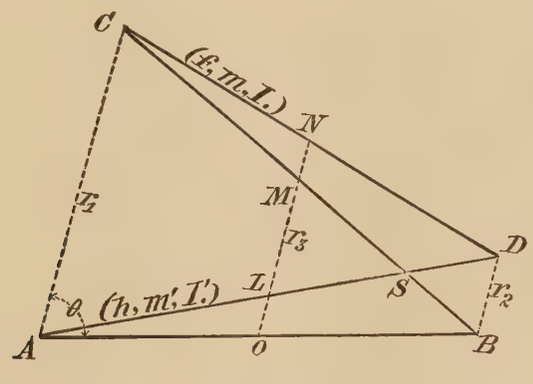

Fig. 11.

Let $AB = CD = f$, $AD = BC = h$, let mass of AB = mass of $CD = m$, mass of AD = mass of $BC = m'$, and let O be the middle point of AB, and θ the angle CAB in any position of the system.

The triangles ADC and CBA having their sides equal are equal in every respect, and therefore AC is parallel to BD, also $AS = SC$, and $BS = SD$. If OL be drawn parallel to AC it passes through L, M, and N, the middle points of the bars AD, BC, CD.

Let $AC = r_1$, $BD = r_2$, $ON = r_3$, and $OG = r$, where G is the centre of inertia of the three bars AD, BC, and CD; then we have $r_1 - r_2 = 2f \cos \theta$,

and, since the quadrilateral is inscribable in a circle, $r_1 r_2 = h^2 - f^2$. From these we obtain

$$r_1 + r_2 = 2\sqrt{(h^2 - f^2 \sin^2\theta)}, \quad r_3 = \sqrt{(h^2 - f^2 \sin^2\theta)},$$

$$r_1 = r_3 + f\cos\theta, \quad r_2 = r_3 - f\cos\theta,$$

$$r = \frac{m + m'}{2m' + m}\sqrt{(h^2 - f^2 \sin^2\theta)}.$$

If we put $\dfrac{m + m'}{2m' + m} = \nu$, we have, if y be the ordinate of G the centre of inertia, $y = \nu \sin\theta \sqrt{(h^2 - f^2 \sin^2\theta)}$.

The positions of equilibrium are given by the equation $\dfrac{dy}{d\theta} = 0$, *i. e.* by the equation

$$\frac{\nu \cos\theta}{\sqrt{(h^2 - f^2 \sin^2\theta)}}(h^2 - 2f^2 \sin^2\theta) = 0,$$

or by the two equations

$$\cos\theta = 0, \quad \sin^2\theta = \frac{h^2}{2f^2}.$$

If $h^2 > 2f^2$ there is only one position of equilibrium, but if $h^2 < 2f^2$ there are three given by the equations

$$\theta = \frac{\pi}{2}, \quad \theta = \sin^{-1}\frac{h}{f\sqrt{2}}, \quad \theta = \left(\pi - \sin^{-1}\frac{h}{f\sqrt{2}}\right).$$

When $\cos\theta = 0$, we have

$$\frac{d^2y}{d\theta^2} = \frac{-\nu(h^2 - 2f^2)}{\sqrt{(h^2 - f^2)}},$$

which is positive if $h^2 < 2f^2$. When $\sin^2\theta = \dfrac{h^2}{2f^2}$ we have

$$\frac{d^2y}{d\theta^2} = \frac{-4\nu f^2 \cos^2\theta \sin\theta}{\sqrt{(h^2 - f^2 \sin^2\theta)}},$$

which is negative. Hence, if $h^2 < 2f^2$, there are three positions of equi-

librium, one stable and two unstable. If $h^2 > 2f^2$ there is only one position of equilibrium which is unstable, because, in this case, if $\cos\theta = 0$, we find that $\frac{d^2y}{d\theta^2}$ is negative.

If the system be such as to admit of a position of stable equilibrium, the small oscillations of the system about that position may be determined in the following manner:—

Let I = moment of inertia of CD round N,
I' = moment of inertia of AD round A,
= moment of inertia of BC round B,
ω_1 = angular velocity of AD,
ω_2 = angular velocity of BC,
ω_3 = angular velocity of CD,
v = velocity of point N;

then, if T be the kinetic energy of the system,

$$2T = I'(\omega_1^2 + \omega_2^2) + I\omega_3^2 + mv^2.$$

When the system is in the position of stable equilibrium,

$$\omega_1 = \dot{\theta}, \quad \omega_2 = \dot{\theta}, \quad \omega_3 = 2\dot{\theta}, \quad v = \dot{\theta}\sqrt{(h^2 - f^2)},$$

as may be proved in the following manner:—

At any time D is describing a circle round A as centre, and C a circle round B as centre. Hence S is the instantaneous centre of rotation of CD (Williamson and Tarleton, "Dynamics," Arts. 219, 221), and we have

$$AD\omega_1 = SD\omega_3, \quad BC\omega_2 = SC\omega_3;$$

therefore, when $\theta = \frac{\pi}{2}$, we have $\omega_1 = \omega_2 = \frac{1}{2}\omega_3$.

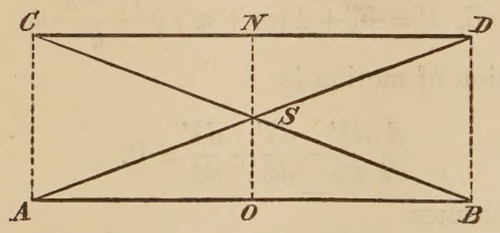

Fig. 12.

Again, since S is the instantaneous centre of rotation, the motion of N

is perpendicular to ON, and therefore $SN.\omega_3 = ON.\dot{\theta}$, whence $\omega_3 = 2\dot{\theta}$. Substituting for ω_1, ω_2, ω_3, and v, and putting $h^2 - f^2 = k^2$, we obtain $(2I' + 4I + mk^2)\,\dot{\theta}^2$ as the value of $2T$ in the position of equilibrium. If the system be performing small oscillations about this position we may put $\theta = \frac{\pi}{2} + \phi$, where ϕ is small. In general, $2T$ is of the form $\dot{\theta}^2 f(\theta)$, and when $\theta = \frac{\pi}{2} + \phi$, we have

$$2T = \dot{\theta}^2 \left\{ f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\phi + \&c. \right\}.$$

In the position of equilibrium $\phi = 0$, and $2T = \dot{\theta}^2 f\left(\frac{\pi}{2}\right)$. Hence we get $f\left(\frac{\pi}{2}\right) = 2I' + 4I + mk^2$.

Now, when ϕ and $\dot{\theta}$ are small, we have, approximately, $2T = \dot{\theta}^2 f\left(\frac{\pi}{2}\right)$, and in the case of small oscillations,

$$2T = (2I' + 4I + mk^2)\,\dot{\theta}^2 = (2I' + 4I + mk^2)\dot{\phi}^2.$$

The potential energy V is, in general, $(m + 2m')\,gy$. If V_0 be the value of V when $\theta = \frac{\pi}{2}$, we have

$$V = V_0 + \left(\frac{dV}{d\theta}\right)_0 \phi + \tfrac{1}{2}\left(\frac{d^2V}{d\theta^2}\right)\phi^2 + \&c.;$$

now

$$\left(\frac{d^2V}{d\theta^2}\right)_0 = (m + 2m')\,g\left(\frac{d^2y}{d\theta^2}\right)_0 = (m + 2m')\,gv\frac{(2f^2 - h^2)}{\sqrt{(h^2 - f^2)}} = (m + m')\,g\frac{f^2 - k^2}{k};$$

hence, when ϕ is small, $V = V_0 + \frac{1}{2}(m + m')\,g\frac{f^2 - k^2}{k}\phi^2$.

Legrange's equation of motion is

$$\frac{d}{dt}\frac{dT}{d\phi} - \frac{dT}{d\phi} + \frac{dV}{d\phi} = 0,$$

which in this case becomes

$$(4I + 2I' + mk^2)\frac{d^2\phi}{dt^2} + (m + m')\,g\frac{f^2 - k^2}{k}\phi = 0.$$

Hence the small oscillations of the system are isochronous with those of a simple pendulum whose length l is given by the equation

$$l = \frac{4I + 2I' + mk^2}{(m + m')(f^2 - k^2)} \cdot k.$$

Since $AS + SB = AD = h$, it is plain that in the motion of the system the point S describes an ellipse whose pedal is described by the middle point of ON. If $2a$ and $2b$ be the axes of the ellipse, and $2c$ the distance between its focii, we have $2a = h$, $2b = k$, $2c = f$. The condition that stable equilibrium should be possible is $a^2 > 2b^2$, and the length l of the isochronous simple pendulum is given by the equation

$$l = \frac{2I + I' + 2mb^2}{(m + m')(a^2 - 2b^2)} \cdot b.$$

END OF VOLUME XXIX.

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